On maintenance modeling by first passage times of stochastic processes

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Contents of talk

- Introduction to Wiener processes and gamma processes
- Applications to maintenance using threshold models
- Prediction of future failures
- Case study on wind turbines
- Concluding remarks

This is joint work with earlier and present students and researchers in Trondheim

First passage time models

Some typical examples from the literature:

- *Wiener processes:*
  - Whitmore, The Statistician 1986: First-passage-time models for duration data: regression structures and competing risks.
  - Whitmore and Schenkelberg, LIDA 1997: Modelling accelerated degradation data using Wiener diffusion with a time scale transformation.
  - Whitmore, Crowder and Lawless, LIDA 1998: Failure inference from a marker process based on a bivariate Wiener model.
- *Gamma processes:*
  - Lawless and Crowder, LIDA 2004: Covariates and random effects in a gamma process model with application to degradation and failure gamma process modeling.
  - van Noortwijk, RESS 2009: A survey of the application of gamma processes in maintenance. (Comprehensive treatment!)

Wiener process

A stochastic process $\{W(t), t \geq 0\}$ is a *Wiener process* with drift coefficient ν and variance parameter σ^2 if

1. $W(0) = 0$,
2. $\{W(t), t \geq 0\}$ has stationary and independent increments,
3. for every $0 < s < t$, $W(t) - W(s)$ is normally distributed with mean $\nu(t-s)$ and variance $\sigma^2(t-s)$.

Inverse Gaussian distribution

A special feature that makes the Wiener process mathematically tractable is that the first passage time to a level $a > 0$ is inverse Gaussian distributed with density as given below. Note that if we redefine ν as ν/σ and a as a/σ, then we may assume that $\sigma^2 = 1$.

$$f(t;\nu,a) = \frac{a}{\sqrt{2\pi}} t^{-\frac{3}{2}} \exp\left\{-\frac{(a-\nu t)^2}{2t}\right\}, t > 0,$$

We denote this distribution by $\mathrm{IG}(\nu, a)$, the inverse Gaussian distribution with parameters ν and a. The corresponding survival function is given by

$$S(t;\nu,a) = \Phi\left(\frac{a-\nu t}{\sqrt{t}}\right) - e^{2a\nu}\Phi\left(\frac{-a-\nu t}{\sqrt{t}}\right). \tag{1}$$

where Φ is the standard normal cumulative distribution function.

Gamma process

A stochastic process $\{X(t), t \geq 0\}$ is a *(non-stationary) gamma process* with (non-decreasing) shape function $v(t)$ and scale parameter $u > 0$ if

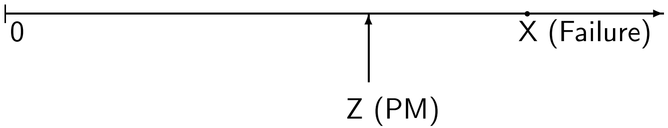
1. $X(0) = 0$,
2. $\{X(t), t \geq 0\}$ has independent increments,
3. for every $0 < s < t$, $X(t) - X(s)$ is gamma distributed with shape parameter $v(t) - v(s)$ and scale parameter u.

Special property: $X(t)$ is non-decreasing.

A stationary gamma process has $v(t) = vt$ for a constant v.

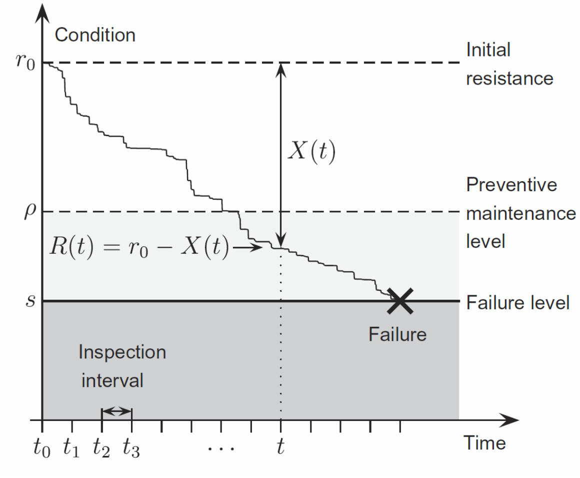
First application:
Dependent competing risks involving failure and PM

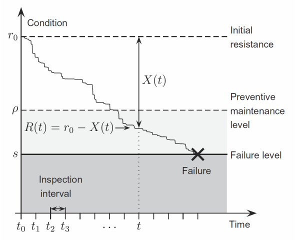
Competing risks between failure (X) *and PM* (Z)



Threshold models using gamma process

$X(t)$ is a degradation process given as a gamma process.



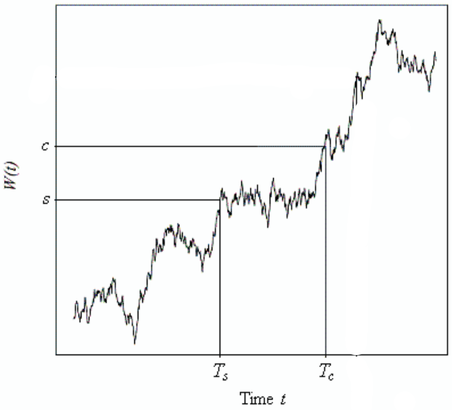


Decision variables:

- inspection interval
- PM level (random)
- state is observed only at inspections
- item is renewed by PM or corrective maintenance according to which threshold is crossed at inspection

Threshold models using Wiener processes

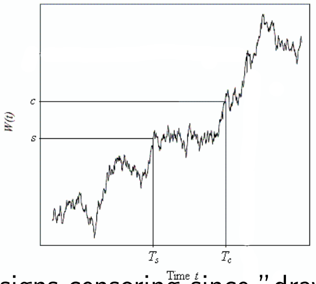
Let c be the threshold for failure, while s is a threshold for a possible PM



BL and Skogsrud (IIE Transactions 2008)

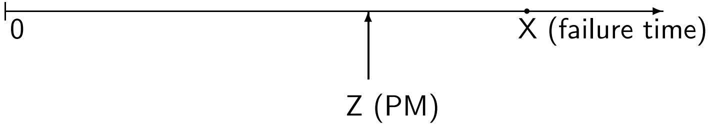
Wiener process $W(t)$ with positive drift, ν, unit variance $\sigma^2 = 1$
T_a = hitting time of $a > 0$.

- Let $X = T_c$
- and $Z = \begin{cases} T_s, & \text{with probability } q \text{, independent of } X \\ \text{not observed with probability } 1-q \end{cases}$



Remarks:

- Z is a random signs censoring since "draw" made at T_s is independent of X
- X has an inverse Gaussian distribution



Definition:

- The event $\{Z < X\}$ is independent of X

Motivation:

- Suppose the item emits some *warning* of emerging failure, *prior to failure.*
  If warning signal is observed, then the item will be preventively maintained at some time Z.

  If the event of observing the signal is independent of the item's potential failure time, then *random signs censoring* is appropriate.

Identifiability in random signs censoring

Random signs censoring is equivalent to

$$\tilde{S}_X(x) \equiv P(X > x | X < Z) = P(X > x) \equiv S_X(x) \text{ for all } x \quad (*)$$

It follows that the marginal distribution of X is identifiable from competing risks data under random signs censoring.

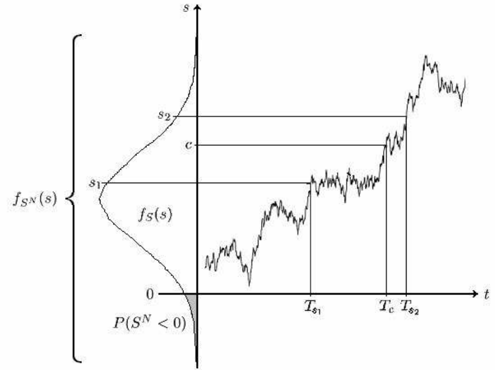
But the condition (*) may look very unreasonable for many applications.

(Still it holds in the Wiener-process model by BL and Skogsrud.)

Extension of model by BL and Skogsrud: Random PM level S

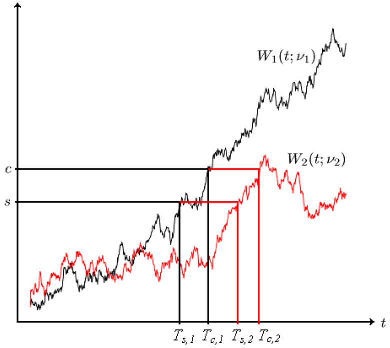
Let $W(t)$ be as before, let $S \sim N(\mu_S, \sigma_S^2)$, truncated on $(0, \infty)$, independent of $W(\cdot)$

- $X = T_c$, $Z = T_S$
- *Random signs censoring* holds since
  - $Z < X \Leftrightarrow S < c$ and S and T_c are independent



Extension of model by BL and Skogsrud: Random drift

- Reconsider basic model with fixed c and s.
- Let drift ν vary randomly across units according to some distribution. (This is similar to frailty for models defined by conditional intensities).
- The drift then needs to be integrated out in the likelihood.



Likelihood functions for the BL and Skogsrud models

Basic model:

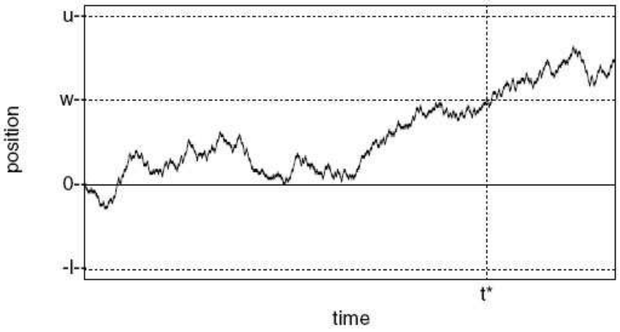
- Contribution from an observed $X = x$: $(1-q)f(x;\nu,c)$
- Contribution from an observed $Z = z$: $q\ f(x;\nu,s)$
- Contribution from a censored observation τ: $(1-q)S(\tau;\nu,c) + q\ S(\tau;\nu,s)$

Random S:

- Contribution from an observed $X = x$: $(1-F_S(c))f(x;\nu,c)$
- Contribution from an observed $Z = z$: $\int_0^c f_S(s)f(z;\nu,s)ds$ (now are T_S and S dependent).
- Contribution from a censored observation τ: $(1-F_S(c))S(\tau,\nu,c) + \int_0^c f_S(s)S(\tau;\nu,s)ds$

Horrocks and Thompson (LIDA 2004): Hospital stays

Similar model for another type of application.



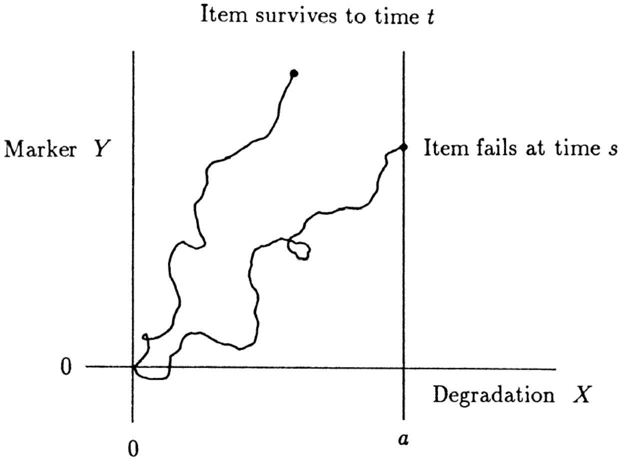
$$u = \text{healthy discharge from hospital}$$
$$-l = \text{death}$$
$$w = \text{transfer to other institution, with probability } p$$

Horrocks and Thompson

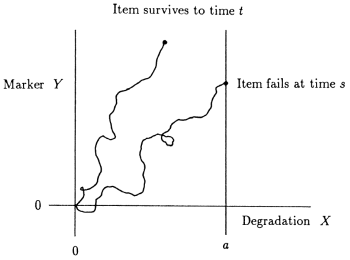
"In some situations, individuals will become eligible for transfer to another acute care institution once their health status reaches a moderate level, w, where $0 < w < u$. To develop a model, we assume that once this level is achieved, an individual will be transferred with probability p. We make the simplifying assumption that only the first visit of the health level process to w potentially triggers a transfer."

Predicting future failures (Whitmore, Crowder, Lawless LIDA 1998)

$(X(t), Y(t))$ is a bivariate Wiener-process, where $X(t)$ is an unobservable degradation process, while $Y(t)$ is an observed marker process that is correlated with the true degradation process.



Predictive inference (Whitmore et al.)



Two types of predictive inference that exploit the observed marker information are considered. The second is of primary interest but the first is needed in order to address the second.

1. Prediction of the degradation level $X(t)$ of a surviving item at time t, given that its marker level at that time is $Y(t) = y$.
2. Prediction of the future failure time S of an item that is surviving at time t, given that its marker level at time t is $Y(t) = y$.

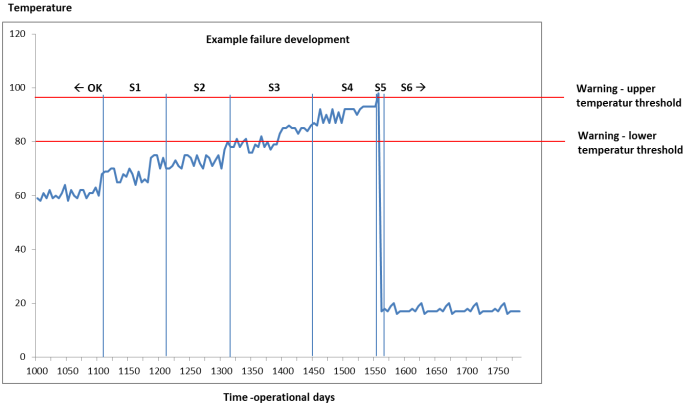
Case study: Failure prediction for bearing in offshore wind turbine based on condition monitoring

Suggested for TCI modeling (yellow)

Health indicator	Detectable at stage 1	2	3	4	5	6	CM method
Particles in oil	X	X	X	X			Oil analysis
Vibrations		X	X	X	X		Machine monitoring
Abnormal noise				X	X		**Human senses**
Temperature variations		X	X	X	X	X	Machine monitoring
Visible signs		X	X	X	X	X	Inspection (visual or tools)
	Impurities in oil	Mechanical wear	Micropitting, pitting	Chipping	Bearing breakdown	Turbine shut down/damage	

ECHNICAL CONDITION MATRIX

Temperature development of bearing



Probabilistic modeling of failure development

- $Y(t)$ is (observed) temperature at time t (days).
- The aim is to predict the future behavior of temperature from observation of $Y(t)$ from time $t = 0$ to some given time t_0.
- The object of main interest for prediction is the time when the temperature exceeds some threshold a.

A latent Markov process

- Development in stages $1 - 2 - 3 - \cdots$ is modeled as a latent discrete state process $S(t)$, with $S(0) = 0$ and $S(t)$ being stage number at time t.
- Assume $S(t)$ is a (non-decreasing) continuous time Markov process with time-homogeneous transition rates.
- The probability mechanism of the temperature process $Y(t)$ depends on the state $S(t)$.

The Wiener process based model

- The stochastic process $Y(t)$ is a Wiener process under each "regime", but parameters may change when $S(t)$ changes.
- Natural to assume that the drift parameter ν equals 0 when $S(t) = 0$ ("normal conditions").
- Under the failure development through Stages 1 and above, the drift is assumed to be positive, with values ν_i when $S(t) = i$. ν_i increases with i. Likewise, the variance parameter σ^2 may depend on the state, but may also be kept constant.

Special case: The Wiener process with a single change point

Assumptions:

- $Y(t)$ follows a Wiener process with $Y(0) = 60$
- Until time τ (the time of entrance to Stage 1) there is no drift, $\nu = 0$.
- From time τ on, there is a positive drift ν.

T is the time when $Y(t)$ crosses the given level $a > 0$. We make the simplifying assumption that

- T is always larger than τ. More precisely,

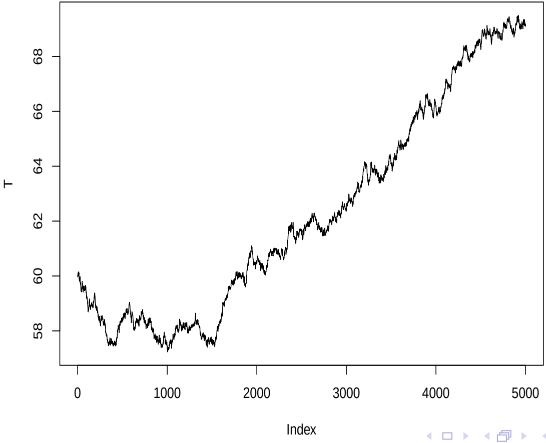
$$T = \inf\{t > \tau | W(t) \geq a\}$$

Then conditional on τ and $W(\tau)$, $T - \tau$ has an Inverse Gaussian distribution as already described. This corresponds to the time of hitting the threshold $a - W(\tau)$ for a Wiener process with drift ν, starting at state 0. Thus the conditional expected value of T will be $\tau + (a - W(\tau))/\nu$.

Simulation

Let $\tau = 1500$, $\sigma = 0.05$, $\nu = 0.003$. If $a = 65$, say, then the expected value of T (crossing 65 degrees) is

$$1500 + \frac{65 - 58}{0.003} = 3833$$



- Assume it is *known by expereince* that $\sigma = 0.05, \ \nu = 0.003$.
- Assume that the process has been observed from time 0 to time t_0. (We consider $t_0 = 2500$, 3000 or 5000)

Problem considered: To estimate τ based on the observation up to t_0.

From the estimate $\hat{\tau}$, say, we can e.g. estimate the expected time of crossing the threshold a to be

$$\hat{\tau} + (a - W(\hat{\tau}))/\nu,$$

A discrete time solution

We use a Bayes approach. Assume (for simplicity) that the temperature $Y(i)$ is observed at discrete time points, here *days*, $i = 1, 2, \ldots, n$.

The differences $X_i = Y(i+1) - Y(i)$ are independent and normally distributed.

Now by the assumptions, for an unknown positive integer time τ:

- $X_1, \ldots, X_{\tau-1}$ are $N(0, \sigma^2)$
- while $X_\tau, X_{\tau+1}, \ldots, X_n$ are $N(\nu, \sigma^2)$.

Following Shiryaev (1963) τ is given a geometric prior with

$$\pi(\tau) = q(1-q)^{\tau-1} \text{ for } \tau = 1, 2, \ldots \tag{2}$$

where q is a given number. Note that then the prior expectation of τ is $1/q$. q has the reasonable interpretation as the probability of a switching happening at any given day.

The posterior distribution

First, the likelihood function for our data is

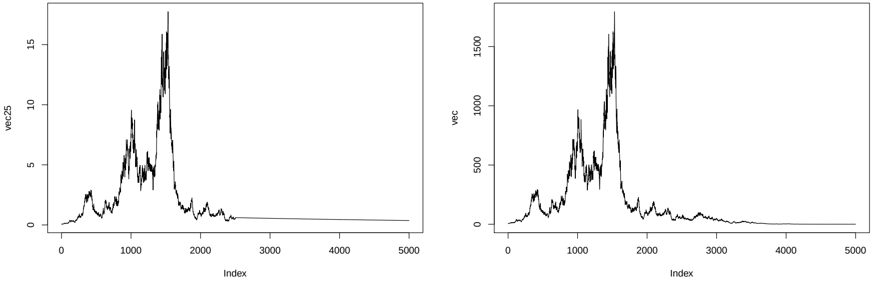
$$L(x_1,\ldots,x_n|\tau) \propto \begin{cases} \exp\{-\frac{1}{2\sigma^2}(\sum_{i=1}^{\tau-1} x_i^2 + \sum_{i=\tau}^{n}(x_i-\nu)^2)\} & \text{if } \tau \le n \\ \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right\} & \text{if } \tau > n \end{cases}$$

The posterior for τ is proportional to $\pi(\tau)L(x_1,\ldots,x_n|\tau)$.

Multiplying L by $\exp\left\{(1/2\sigma^2)\sum_{i=1}^{n} x_i^2\right\}$ we hence get the posterior distribution for τ on the form

$$\pi(\tau|x_1,\ldots,x_n) \propto q(1-q)^{\tau-1} \cdot \begin{cases} \exp\{\frac{1}{2\sigma^2}[2\nu\sum_{i=\tau}^{n} x_i - (n-\tau+1)\nu^2]\} \\ 1 \end{cases}$$

Posterior distributions



Posterior distributions (non-normalized) for τ, at times $n = 2500$ and $n = 5000$. Prior for τ is $q = 1/5000$.

Mean values for the posteriors are respectively 2275 and 1380 for $n = 2500$ and $n = 5000$, while maximum posterior value for τ is close to the true value 1500 for both,

The continuous time solution

Suppose now that the time parameter t is continuous and that $\tau > 0$ is a continuous parameter.
This case can be seen as the limiting case when the discrete time unit tends to 0;

- The geometric prior distribution for τ becomes the exponential distribution,

$$\pi(\tau) = \lambda e^{-\lambda\tau} \text{ for } \tau > 0$$

  where the expected value of the prior distribution is now $1/\lambda$.

- The posterior distribution for an observation of $W(t)$ from $t = 0$ to $t = t_0$ is seen to be of the form

$$\pi(\tau | W(t), 0 \leq t \leq t_0)$$

$$\propto \lambda e^{-\lambda\tau} \cdot \begin{cases} \exp\{\frac{1}{2\sigma^2}[2\nu(W(t_0) - W(\tau)) - (t_0 - \tau)\nu^2]\} & \text{if } \tau \leq t_0 \\ 1 & \text{if } \tau > t_0 \end{cases}$$

Markov Chain Monte Carlo (MCMC) solution

- For the discrete time case considered above, we may need a normalization of the posterior distribution to finding the expected posterior value for τ.
- Alternatively, one may use an MCMC solution by for example the Metropolis-Hastings method. There are several possible ways here, for example using an independence sampler which draws proposals from the prior distribution. (More efficient algorithms can of course be thought of).
- Suppose now that also ν is an unknown parameter. The posterior distribution for (τ, ν) is then given by replacing the prior distribution for τ (or the continuous time version by the joint prior of (τ, ν), which might be the product of their marginal priors if they are assumed independent.
- If also σ^2 is assumed unknown, then the given likelihood function shows that the given expressions for the posterior distribution can not be used. Still MCMC is possible. It may, however, in practice be reasonable to assume that the value of

A similar Gamma process approach

Fouladirad et al. (RESS 2008) use a gamma process model. The problem that is studied is that of detecting the point in time, τ, where the deterioration process reaches a more severe level, given by new values for the parameters of the gamma process. (The article considers the detection of the change as a means for controlling condition based maintenance).

Some concluding remarks

- It seems that for practical purposes the gamma process is more useful since it is non-decreasing. Hence it can be used to model cumulative damage. The Wiener-process on the other hand fluctuates both up and down with time. This may be reasonable for some applications, but not for other. One good reason to use the Wiener process is though that it has attractive mathematical properties.
- The Wiener process becomes more flexible if a time-transformation is introduced, so that one uses $X(t) = W(\Lambda(t))$ as a mdel (e.g. Whitmore and Schenkelberg LIDA 1997; Doksum and Høyland Tecnometrics 1992).
- The multivariate Wiener process is easy to describe and may be a useful model when there are several components under consideration at the same time (for example in the wind turbine case study). See also the cited paper by Whitmore, Crowder and Lawless LIDA 1998.

Continued

- Switching models like the one suggested in the wind turbine case study are common in the literature. A reliability application is given by Chiquet, Eid and Limnios RESS 2008 ("Modelling and estimating the reliability of stochastic dynamical systems with Markovian switching).